Appendix A

Frequency doubling 922 nm

A.1 Non-linear crystal

Potassium Niobate ($KNbO_3$), which has a high effective non-linear coefficient, is used for second-harmonic generation to blue wavelengths. Here, 922 nm light is frequency-doubled to 461 nm to address the principal transition in strontium for laser cooling. The crystal dimension along the length of the home-made resonator is 5 mm and has a plane reflecting surface. The other mirror, acting as the input coupler, has a radius of curvature (ROC) of 25 mm (see Fig. A.2). Below is a cartoon of the doubling cavity with relevant components [1].

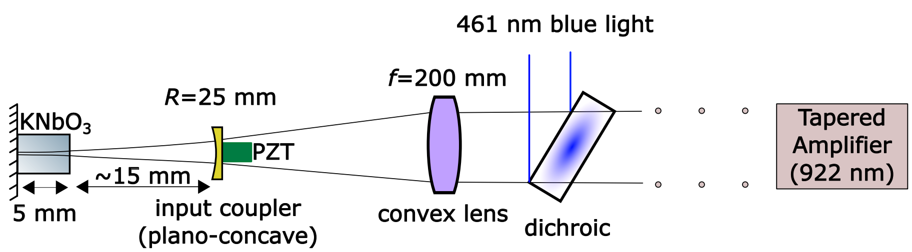

Figure A.1: Layout of the components for doubling 922 nm light from a tapered amplifier (TA), seeded by the 922 nm primary laser which is frequency-locked to strontium atoms.

A.2 Resonator

One of the important aspects for efficient conversion is the intensity of light within the doubling crystal. Hence, the incoming 922 nm light needs to be mode-

[1] In the near future we will be replacing the TA light with light from a long fiber sourced from the primary laser in the plasma lab.

matched to the fundamental spatial mode of the cavity (also called TEM_{00}) which, maximises the intensity of 922 nm light on the non-linear crystal. Typically, this amounts to matching the size and position of the waist of the incoming beam to the natural waist size and position of the fundamental cavity mode. Worse alignment leads to coupling to the non-fundamental (higher order) spatial modes, thereby reducing doubling efficiency. Matrix algebra, as described in [1], can be used to calculate the natural waist size and waist position of such a resonator. The elements of the matrix are dependent on various parameters of the resnator such as its length, refractive index of the media, and the radius of curvature of the mirrors that form the cavity.

A.2.1 Expression for the fundamental spatial mode

The wave equation for light, parameterized by x, y, z, t, where $\mathrm{U}(x, y, z, t)$ denote the instantaneous field amplitude, is given by:

$$\nabla^2 U = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}U \tag{A.1}$$

For a monochromatic radiation, as is the case for our laser ($\leq$100 MHz linewidth), the general wave equation simplifies to a scalar wave equation with $U(x, y, z; t) = U(x, y, z)e^{-i\omega t}$. Substituting this into eq. (A.1), we get:

$$\nabla^2 U + k^2 U = 0, \tag{A.2}$$

where, k is the magnitude of the wavevector given by the relation $k = \frac{2\pi}{\lambda}$.

For a cylindrically symmetric beam travelling in the $z-$direction (we take this axis to be axis of propagation throughout the report),

$$U(x, y, z; t) = \psi(x, y, z)e^{-ikz} \tag{A.3}$$

The scalar wave equation modifies to:

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial \psi}{\partial r} - i2k\frac{\partial \psi}{\partial z} = 0 \tag{A.4}$$

The above equation has a multiple solutions corresponding to different spatial (transverse) and longitudnal modes (frequency) [2]. The simplest spatial mode, i.e. TEM_{00}, is given by:

$$\psi(r, z) = e^{-iP(z) - i\frac{kr^2}{2q(z)}} \tag{A.5}$$

where, $r^2 = x^2 + y^2$. Without loss of generality, $P(z)$ and $q(z)$ are complex numbers. Let $q(z)$ be written in terms of a real and imaginary parameter, given by:

$$\frac{1}{q(z)} = \frac{1}{R(z)} - \frac{i\lambda}{\pi s(z)^2}, \tag{A.6}$$

[2] One can use Green's function's approach to find the solution of the wave equation.

where, $s(z) = \sqrt{s_0^2[1 + (\frac{\lambda z}{\pi s_0^2})^2]}$ is the spot size of the gaussian beam. The waist size is denoted as s_0. $R(z)$ gives the radius of curvature of the wavefront that intersects the propagation axis at z. $P(z)$ is the complex phase and obeys the relation,

$$P'(z) = -\frac{i}{q(z)}, \tag{A.7}$$

where,' $'$ ' is the first-order derivative with respect to z.

Physical interpretation of $s(z)$

The intensity is given by :

$$I = \frac{1}{2}\epsilon_0 c|\psi(r,z)|^2 \propto e^{-\frac{kr^2\lambda}{\pi s^2}} (= e^{\frac{-2r^2}{s^2}}), \tag{A.8}$$

where, the proportionality relation is obtained by substituting (A.6) in (A.5). Thus, the spotsize $[s(z)]$ of a gaussian beam is defined as the distance from the beam axis (at position z) where the intensity drops to $1/e^2(\approx 13.5\%)$ of the maximum value.

Before we go about determining the natural waist size and position of our cavity, here is a brief review of the ABCD matrix algebra for the transformations of the complex beam parameter $q(z)$.

Propagation of light through media of different refractive index

From the ABCD matrix theory[1], the transfer matrix for light travelling through a medium of length d_1 and refractive index n_1, is given by:

$$T_{\text{distance}} = \begin{pmatrix} 1 & d_1/n_1 \\ 0 & 1 \end{pmatrix}. \tag{A.9}$$

Therefore, light travelling through two media for different lengths and different refractive indices has a transfer matrix given by,

$$T_{\text{distance}} = \begin{pmatrix} 1 & d_1/n_1 + d_2/n_2 \\ 0 & 1 \end{pmatrix}. \tag{A.10}$$

Thus, an effective transfer matrix can be defined where, the light travels through an effective single media (air) of length $d_1/n_1 + d_2/n_2$.

Propagation of light through lens

The ABCD matrix for light travelling through a lens of focal length f is given by,

$$T_{\text{lens}} = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}. \tag{A.11}$$

Transfer matrix for doubling cavity

For the case of the doubling cavity, the effective ABCD matrix is obtained by matrix multiplication of the individual transfer matrices. The order of the multiplication is determined by the order in which the light ray passes through each of the optical elements.

For a single pass through the cavity, the light beam travels a distance d_{cry} through the crystal, travels a distance d_{air} in air, reflects from the input coupler, travels back through air and then ends the cycle by traveling through the crystal once again.
Let T be the transfer matrix for such a single pass. It is then given by:

$$[T]_{mn} = [T_{crystal}]_{ma} \cdot [T_{air}]_{ab} \cdot [T_{coupler}]_{bc} \cdot [T_{air}]_{cd} \cdot [T_{crystal}]_{dn} \tag{A.12}$$

where, '.' represents matrix multiplication. The transformation of beam parameter, $q(z)$, is given by:

$$q_2 = \frac{Aq_1 + B}{Cq_1 + D}, \tag{A.13}$$

where, $A\,, B\,, C\,, D$ are elements of T.

To realize a confocal cavity with stable modes, the beam parameter transformation should be repeatable for infinite cycles. It suffices to say that the beam parameter $q(z)$ after a single pass through the elements in the cavity remains unchanged, i.e., $q_2 = q_1$.

A.2.2 Confirming radius of curvature of input coupler

Aaron Saenz and Joshua Hill's[3] theses contain information regarding the parameters of the 922 nm doubling cavity. The length of the crystal quoted is 5 mm, whose refractive index (n) is 2.28. The radius of curvature of the input coupler is 25 mm. The focal length for the cavity is $f = R/2 = 12.5$ mm. The air medium inside the cavity is around 15 mm. The effective cavity length with only air is then given by,

$$d = 5\,\text{mm}/2.28 + 15\,\text{mm} \approx 17.2\,\text{mm}. \tag{A.14}$$

However, the input coupler acts as a plano-concave lens for light outside the cavity whose focal length is -50 mm. The distinction between $f(= R/2)$, which is used for calculating the natural waist of cavity, and its 'lens' focal length, which is used to calculate the virtual waist for the incoupling light, needs to be clear to avoid error.

A.2.3 Source of error in [2]

Figure 2.4 of reference [2] shows a cavity which is larger in size than it actually is. Figure A.3 of this report shows the source of error. This leads to a calculation of natural waist size and position which is different from the actual natural waist in the experiment. This calculation is redone in the next section.

[3] https://ultracold.rice.edu/research.shtml

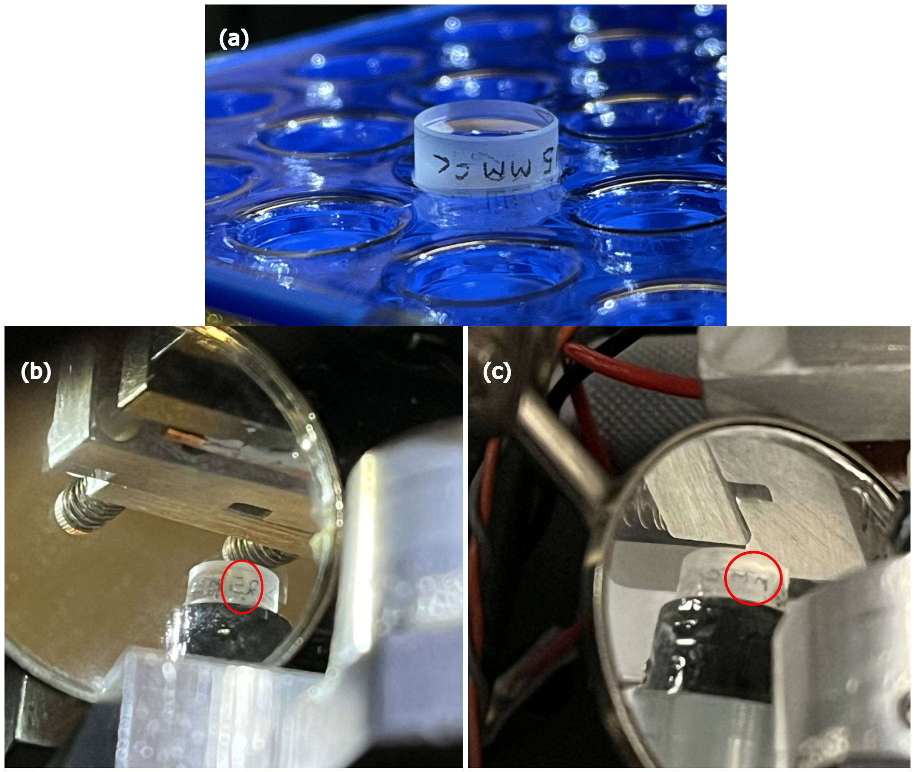

Figure A.2: (a) A stock coupler with specifications written on the surface. (b) and (c) Input coupler for the 922 nm doubling cavity with similar specifications. From the image we conclude that the radius of curvature is 25 mm. The red circles point to the significant values (inverted for being a mirror image).

A.2.4 Calculating "natural" and "virtual" waist

Natural waist is defined as the waist of the fundamental mode of the confocal cavity. The virtual waist is the waist of the beam that is being coupled to the cavity when viewed without the input coupler. The input coupler acts as a concave lens of $f = -50$ mm and shifts the waist position of the coupling beam towards the crystal.

Natural waist

The ABCD matrix for our cavity after the matrix multiplications is given by a general expression as shown in Fig. A.4.

Substituting parameters for the cavity, the transfer matrix for our doubling cavity is:

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} -0.375 & 0.01074 \\ -80 & -0.375 \end{pmatrix}. \tag{A.15}$$

(a)

$$L_{eff} = D_{air} + D_{crystal} * n_{crystal} \tag{2.4}$$

$$R = \sqrt{R_1 R_2} \tag{2.5}$$

Where c is the speed of light in vacuum, $D_{air} \approx 15$ mm is the distance in air in the resonator, $D_{crystal} \approx 5$ mm is the crystal length, $n_{crystal} \approx 2.28$ is the index of

(b)

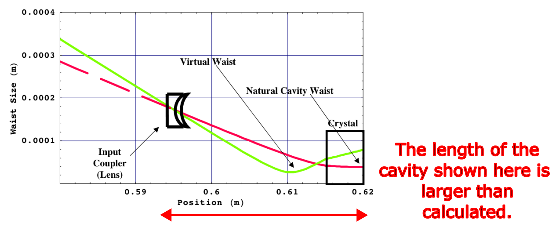

Figure 2.4: **Beam Waists Inside Resonator** Using a Mathematica script (see Appendix B), we can model the natural cavity waist and the virtual waist used for mode matching. The dashed line is the natural cavity waist propagated out of the cavity ignoring the input coupler. The solid line to the left of the input coupler is the beam profile for both the virtual beam and the natural cavity waist propagated out of the cavity with the input coupler acting as a lens.

Figure A.3: (a) The error in calculation could be from this expression (2.4) in [2]. (b) The length of the cavity measured from the figure shown (page 21 of [2]) is larger than what it should be.

All the elements in this matrix have the unit m^{-1}.

The waist size is written in terms of the elements of T as :

$$s_0^2 = \frac{2\lambda B}{\pi\sqrt{4 - (A + D)^2}} \tag{A.16}$$

Eq. (A.16) can be used to plot the variation of waist as a function of d_{air}, as shown in Fig. A.5.

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & \frac{d_{crystal}}{n_{crystal}} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{d_{air}}{n_{air}} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{-2}{R1} & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{d_{air}}{n_{air}} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{d_{crystal}}{n_{crystal}} \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 - \frac{2(\frac{d_{air}}{n_{air}} + \frac{d_{crystal}}{n_{crystal}})}{R1} & \frac{2(d_{crystal}n_{air}+d_{air}n_{crystal})(d_{crystal}n_{air}+n_{crystal}(d_{air}-n_{air}R1))}{n_{air}^2 n_{crystal}^2 R1} \\ -\frac{2}{R1} & 1 - \frac{2d_{air}}{n_{air}R1} - \frac{2d_{crystal}}{n_{crystal}R1} \end{pmatrix} \quad (3.11)$$

where $d_{crystal}$ is the crystal length, n_{air} and $n_{crystal}$ is the air and crystal index of refraction respectively, and d_{air} is the distance from the input coupler to front-face of crystal.

Figure A.4: Equation borrowed from Clayton Simien's thesis for the general expression of the ABCD matrix for a confocal cavity. $R_1 = R$ is the radius of curvature of the input coupler.

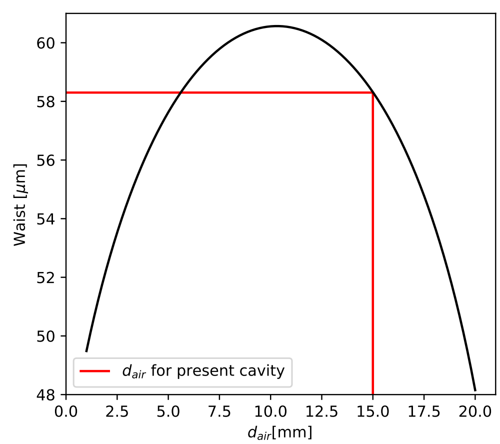

Figure A.5: Waist of the fundamental cavity mode as a function of d_{air}.

Upon simplifying Eq. (A.16), the waist of a confocal cavity can be expressed as:

$$s_0^2 = \frac{\lambda}{\pi}\sqrt{d(R-d)}, \tag{A.17}$$

where, d is the length of the effective cavity and R is the radius of curvature of the input coupler (for our case, $R = 25$ mm). Thus,

$$s_0 = \sqrt{\lambda\sqrt{d(R-d)}/\pi} = \sqrt{922 \times 10^{-9} * 11.6/1000/\pi} = 58.5\,\mu\text{m} \tag{A.18}$$

Virtual waist

The calculation of the virtual waist entails to propagating the fundamental mode outside the cavity with the input coupler acting as a plano-concave lens (with focal length $= -50$ mm). The waist of the lens-transformed beam is called the virtual waist. The virtual waist calculated by both the programs is around 42.8

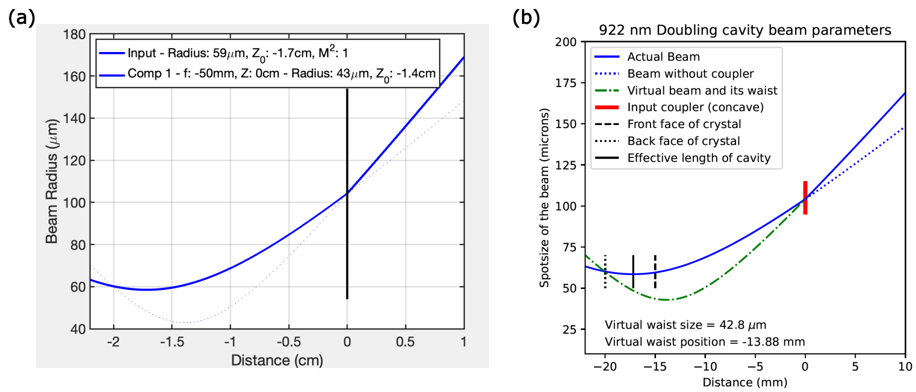

Figure A.6: (a) Virtual waist calculated by Jim Aman's beam propagation program in Matlab. 0 cm is the position of the input coupler. Solid line is the actual beam, whereas dotted lines represent extrapolated features. (b) Virtual waist calculated by SKK's python notebook. There is good agreement implying the python code does not have any errors. 0 mm is the position of the input coupler. (To get a perspective of the effective and actual length of the cavity, see legend.)

microns and the position of such a waist is about 13.88 mm away from the input coupling mirror (or 1.12 mm before the front surface of the crystal, as shown in Fig. A.1).

Bibliography

[1] H. Kogelnik and T. Li, "Laser Beams and Resonators," *Applied Optics*, vol. 5, pp. 1550–1567, Oct. 1966. Publisher: Optica Publishing Group.

[2] A. D. Saenz, "461nm Laser For Studies In Ultracold Neutral Strontium," 2005.