

Correction To the Formula For the Total Frequency Integral of the Ion Velocity Spectrum

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1 Introduction

As of July 2, 2014, the equation we have been using for the total integral of the ion velocity spectrum is incorrect, while the equation for the spectrum itself is correct as far as I know. What is missing is a factor of $1/\sqrt{1+s_0}$, where s_0 is the saturation parameter of the probe beam. A likely explanation for this error is that the original equation was derived with the assumption $s_0 \ll 1$, a fact which has now been forgotten as we have moved to using higher values of s_0 where the assumption is no longer valid. Since the incorrect formula is still in use in some parts of our data analysis, perhaps it can explain some observed discrepancies between the densities calculated from the summed images and the densities calculated from the spectral fits. Also, the missing correction will introduce errors into the plasma sizes calculated from the 2D fits. For reference, the INCORRECT equation is given as 3.7 and 3.8 in Jose Castro's thesis (it is wrong in other documents as well).

This document describes a derivation of the correct expression, and indicates the error in calculated density that might be expected from using the incorrect formula over a range of saturation parameters.

2 Derivation of Correct Result

The equation for the ion spectrum I am going to start with is equation 3.5/3.6 from Jose Castro's thesis. Out in the wild, there are slightly different forms of this equation running around in which camera corrections and spatial integrals are represented in slight different ways, but it does not matter to this argument. So, the starting equation:

$$F(\nu) = C \frac{\gamma_0}{2} \int dz \rho_i(r) \frac{I(r)}{I_{sat}} \int ds \frac{\gamma_0/\gamma_{eff}}{1 + s_0 + \left[\frac{2(\nu-s)}{\gamma_{eff}/2\pi} \right]^2} \times \frac{1}{\sqrt{2\pi}\sigma_D[T_i(r)]} \exp \left\{ -\frac{[s - (\nu_0 + \nu_{exp}^y(r))]^2}{2\sigma_D^2[T_i(r)]} \right\}. \quad (1)$$

where C is OVERALL calibration factor, z is the vertical spatial position along camera line of sight, r is a general position in x, y, z , γ_0 is the natural linewidth

(rad/s), γ_{eff} is $\gamma_0 + \gamma_{laser}$ (rad/s), ρ_i is the ion density, $I(r)$ is the laser intensity, I_{sat} is the saturation intensity, $s_0 = I(r)/I_{sat}$ is the saturation parameter, $\sigma_D = \sqrt{k_B T_i / M}$ is the gaussian width, ν_0 is the center frequency of the ion spectrum (Hz), ν_{exp}^y is the frequency offset due to expansion velocity (Hz), ν is the independent frequency variable (Hz), and s is the dummy variable of integration for the Voigt convolution (Hz).

NOTE that the units on the natural and laser linewidths are in (rad/s) while the units on other frequencies are in Hz. The value we are currently using for γ_0 is $2\pi \times 2.0213 \times 10^7$. The laser linewidth, depending on which laser we are using, is either $2\pi \times 4.7 \times 10^6$ (Toptica SHG-Pro) or $2\pi \times 5.5 \times 10^6$ (doubled Toptica TA-100).

Assume for simplicity that we are at a fixed position in space and looking at an infinitesimally small region so that $I(r)$ and therefore s_0 stay constant. This allows us to treat the frequency integrals independently of the spatial integrals so that we can basically ignore the spatial dependencies for now. To get the total integral under the Voigt profile, we must first integrate in ν to eliminate it from the expression. Integrating over s is then trivial, and we arrive at the total integral under the Voigt profile.

To integrate in ν , start with the general form of a normalized Lorentzian profile centered at 0:

$$L(\nu, \Gamma) = \frac{\Gamma}{\pi} \frac{1}{\nu^2 + \Gamma^2} \quad (2)$$

The integral of this normalized expression from $\nu = -\infty$ to $+\infty$ is equal 1, implying

$$\int_{-\infty}^{+\infty} \frac{1}{\nu^2 + \Gamma^2} = \frac{\pi}{\Gamma} \quad (3)$$

To put the Lorentzian part of equation 1 in this form, multiply by $(\gamma_{eff}^2/16\pi^2)$ / $(\gamma_{eff}^2/16\pi^2)$ to get

$$F(\nu) = C \frac{\gamma_0}{2} \int dz \rho_i(r) \frac{I(r)}{I_{sat}} \int ds \frac{\gamma_0 \gamma_{eff} / 16\pi^2}{\frac{(1+s_0)\gamma_{eff}^2}{16\pi^2} + (\nu - s)^2} \times \frac{1}{\sqrt{2\pi}\sigma_D [T_i(r)]} \exp \left\{ -\frac{[s - (\nu_0 + \nu_{exp}^y(r))]^2}{2\sigma_D^2 [T_i(r)]} \right\}. \quad (4)$$

The Lorentzian parameter Γ for the ion spectrum is therefore given by

$$\Gamma = \frac{\gamma_{eff}}{4\pi} \sqrt{1 + s_0} \quad (5)$$

To integrate from $\nu = -\infty$ to $+\infty$, we can ignore the fact that the Lorentzian part of Equation 4 is centered at the dummy variable s because for finite values of s , the center does not matter for the infinite integral, and for infinite values of s , the gaussian part of the equation is 0. Using Equations 3 and 5 and remembering the assumption of an infinitesimal region so that the spatial and frequency integrals are separable, we see that

$$\int_{-\infty}^{+\infty} F(\nu) d\nu = C \frac{\gamma_0^2}{8\sqrt{1+s_0}} \int dz \rho_i(r) \frac{I(r)}{I_{sat}} \int ds \frac{1}{\sqrt{2\pi\sigma_D[T_i(r)]}} \times \exp\left\{-\frac{[s - (\nu_0 + \nu_{exp}^y(r))]^2}{2\sigma_D^2[T_i(r)]}\right\}. \quad (6)$$

The gaussian part of this equation is already normalized, so, trivially,

$$\int_{-\infty}^{+\infty} F(\nu) d\nu = C \frac{\gamma_0^2}{8\sqrt{1+s_0}} \int dz \rho_i(r) \frac{I(r)}{I_{sat}} \quad (7)$$

This Equation 7 is the correct version of the incorrect Equation 3.7 from Jose Castro's thesis.

3 Expected Errors in Calculated Densities From Using the Incorrect Formula

If the camera is calibrated by using density values extracted from nonlinear fits to the Voigt profile represented by Equation 1, then the expected fractional error in the "summed densities" alternatively calculated from using the incorrect formula for the total integrated spectrum is given very simply by

$$Error = \frac{1 - (1 + s_0)^{1/2}}{(1 + s_0)^{1/2}} = (1 + s_0)^{-1/2} - 1 \quad (8)$$

The plot of error versus saturation parameter s_0 is given in Figure 1. When using the lucite-enclosed 422 doubling cavity built by Clayton Simien as the linearly polarized probe beam, saturation parameters have been around 0.25 or higher. This corresponds to errors of around 10% or higher. In circular polarization, the saturation parameters are higher and so will be the errors.

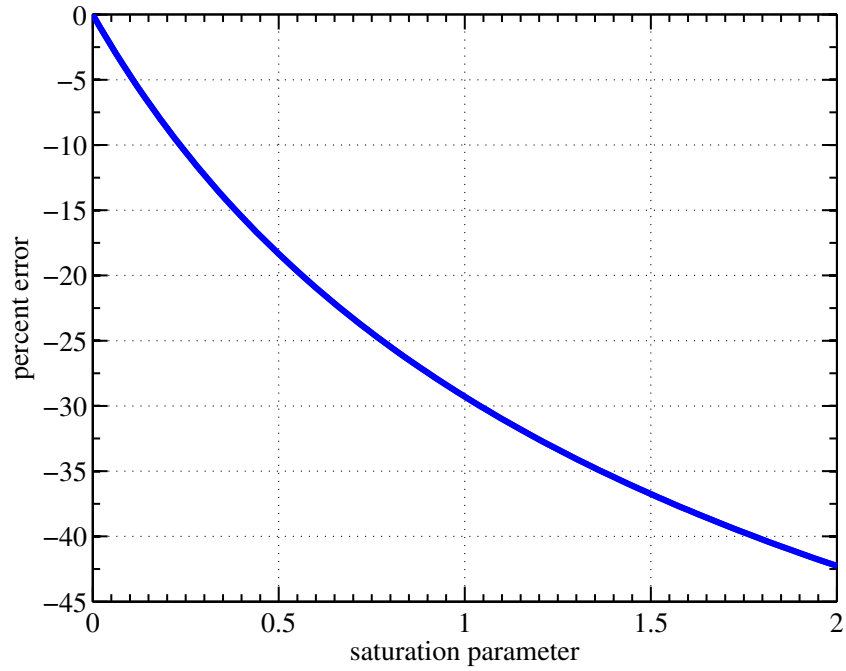


Figure 1: Expected Error In Calculated Densities From Using the Incorrect Formula For the Total Frequency Integral of the Ion Spectrum versus Saturation Parameter

4 Conclusion

Use the correct formula.